Kuramoto model with asymmetric distribution of natural frequencies

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We analyze the Kuramoto model of phase oscillators with natural frequencies distributed according to a unimodal asymmetric function $g(\omega)$. It is obtained that besides a second-, also a first-order phase transition can appear if the distribution of natural frequencies possesses a sufficiently large flat section. It is derived analytically that for the first-order transitions the characteristic exponents describing the order parameter and synchronizing frequency near the critical point are equal to those for the order parameter in the corresponding symmetric case. Stability analysis of the incoherent phase shows that the synchronizing frequency at the onset of synchronization equals the perturbation rotation velocity at the border of stability. The analytic and numerical results are in agreement with numerical simulations.

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I. INTRODUCTION

Synchronization is a phenomenon of coherent behavior of the units of many systems encountered in physics, chemistry, and biology. The Kuramoto model [1,2] is the most successful model describing cooperative phenomena of an infinite number of interacting units represented by phase oscillators. A detailed review of the model and its extensions and applications was published recently [3]. In short, the original Kuramoto model describes synchronization of an infinite number of phase oscillators, interacting with equal all-to-all pairwise coupling. The synchronization is described by an order parameter which is vanishing when the coupling is weak and becomes nonzero when the interaction strength exceeds some critical value.

Here we report on the stationary solution for the Kuramoto model with asymmetric unimodal (single-peaked) distribution function for the natural frequencies of the individual oscillators. The asymmetric case is more natural because any imperfection, however small, can destroy the ideal symmetry. The case with asymmetric distribution has obtained little attention. It was first examined by Sakaguchi and Kuramoto [4] in a more general setting for the interaction between oscillators. They discuss the case of second-order phase transition to coherence and obtain the same characteristic exponents describing the order parameter as in the symmetric case. We are aware of few publications dealing with the asymmetric distribution function for the natural frequencies. The onset of synchronization was studied for the distribution function with exponential tails [5] and for the bimodal distribution exemplified by two δ functions with nonequal strengths [6,7]. Ermentrout [8] considered the stage when all oscillators become synchronized for a piecewise uniform asymmetric distribution consisting of two steps. The case of a first-order phase transition has not been discussed, and therefore we focus our attention on this particular case. For symmetric distributions, first- and second-order phase transitions arise depending on whether the top of the distribution function is flat or not [5,9,10]. The asymmetric distribution also supports first- and second-order transitions, but provides some additional features.

In the following we begin with a description of the model and derive the self-consistent equations for determination of the order parameter. The equations are solved numerically for several examples of asymmetric distribution functions. The results are further supported by stability analysis of the incoherent phase and additionally by numerical solutions of the equations of motion for the interacting oscillators. We conclude with a discussion and an appendix providing proof of the independence of the characteristic exponents for the first-order transitions on the existence of symmetry in the distribution function.

II. MODEL

The Kuramoto model describes the behavior of a large number N of phase oscillators with all-to-all coupling. The phase of every oscillator $\hat{\theta}_i$ evolves according to the equation

$$\dot{\hat{\theta}}_i = \hat{\omega}_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\hat{\theta}_j - \hat{\theta}_i), \tag{1}$$

where the coupling among all pairs of oscillators is equal to K/N. Here we assume that the natural frequencies of the oscillators $\hat{\omega}_i$ are distributed according to some asymmetric unimodal distribution function $\hat{g}(\omega)$, which is nondecreasing on the left from the unique maximum and nonincreasing on the right side. Without any loss in the generality, the maximum can be located at $\omega=0$. For convenience we shall assume, without actually making any restriction, that negative frequencies are more abundant and therefore the mean $\bar{\omega} = \int d\omega\, \hat{g}(\omega)\omega < 0$ is negative.

Following Kuramoto, the degree of coherence of the phases can be expressed through the order parameter

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$$re^{i\hat{\psi}} = \frac{1}{N}\sum_{j=1}^{N} e^{i\hat{\theta}_j}. \tag{2}$$

The synchronization has taken place if the amplitude r has a nonzero value, which happens for couplings stronger than some critical value. Generally, the amplitude and the phase of the order parameter are functions of time, but we shall look for stationary solutions with constant amplitude $r(t)=r$ and phase $\hat{\psi}(t)$, which progresses uniformly in time. Let the phase velocity of the order parameter be equal to some frequency $-\omega_0<0$. It is a quantity we will have to determine later. When $\hat{g}(\omega)$ is symmetric, there is no difference between ω_0, $\bar{\omega}$ and the location of the maximum at $\omega=0$. In a reference frame related to oscillators with frequency $-\omega_0$, the phases of the oscillators are $\theta_i=\hat{\theta}_i+\omega_0 t$ and the phase corresponding to the order parameter is $\psi=\hat{\psi}+\omega_0 t$. In the same frame after an appropriate time shift one can take that $\psi=0$ and the order parameter is identified with its amplitude only:

$$re^{i\psi} = r = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}. \tag{3}$$

Then the coupling in Eq. (1) can be expressed through r as

$$\frac{K}{N}\sum_{j=1}^{N} \sin(\hat{\theta}_j - \hat{\theta}_i) = \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) = K\,\mathrm{Im}\left[\frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}\right] = Kr\sin(\psi - \theta_i). \tag{4}$$

Now the evolution of the phase of each oscillator depends on its coupling with the mean phase

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i) = \omega_i - Kr\sin\theta_i, \tag{5}$$

where the new frequencies are $\omega_i=\hat{\omega}_i+\omega_0$. It is simpler to work in the new reference frame, where instead of the distribution $\hat{g}(\omega)$ the shifted distribution $g(\omega)=\hat{g}(\omega-\omega_0)$ will be used.

For infinite population $N\to\infty$, the oscillators can be described with the probability density function $\rho(\theta,\omega,t)$ of oscillators with intrinsic frequency ω and phase θ at the moment t. The density evolves according to the continuity equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial[\rho v]}{\partial\theta} = \frac{\partial\rho}{\partial t} + \frac{\partial[\rho(\omega - Kr\sin\theta)]}{\partial\theta} = 0. \tag{6}$$

Then the order parameter (3) is defined by an integral:

$$r = \int\int d\theta\, d\omega\, e^{i\theta} g(\omega)\rho(\theta,\omega,t). \tag{7}$$

The assumption that the order parameter r is real implies that

$$r = \int\int d\theta\, d\omega \cos\theta g(\omega)\rho(\theta,\omega,t),$$

$$0 = \int\int d\theta\, d\omega \sin\theta g(\omega)\rho(\theta,\omega,t). \tag{8}$$

The second equation could be identified as the phase balance equation because it ensures vanishing mean phase $\psi=0$. The density ρ is stationary if $\rho v=\text{const}$ [Eq. (6)]. One possibility is $v=0$, which means that the phase of oscillators obeying that condition is

$$\theta = \arcsin\left(\frac{\omega}{Kr}\right). \tag{9}$$

These oscillators are locked to the mean phase and determine the interval of synchronized oscillators. In the original Kuramoto model as well as in the asymmetric case considered here, only such oscillators contribute to the order parameter. The product of the coupling and the order parameter, $\gamma=Kr$, determines the half-width of the synchronized cluster, because only solutions with $|\theta|\leqslant\pi/2$ are stable [3,11].

Oscillators with frequencies outside the interval $-\gamma<\omega<\gamma$ are out of synchrony with the mean phase. Their stationary distribution is $\rho v=C(\omega)=\text{const}$, or

$$\rho(\omega,\theta) = \frac{C(\omega)}{|\omega - \gamma\sin\theta|}. \tag{10}$$

The constant $C(\omega)=\sqrt{\omega^2-\gamma^2}/2\pi$ is determined from the normalization condition $\int d\theta\,\rho(\omega,\theta)=1$. Considering the oscillators with $|\omega|>\gamma$ one remarks [Eq. (10)] that their phases are symmetrically distributed around the phases $\theta=\pi/2$ and $\theta=3\pi/2$ for $0<\theta<\pi$ and $\pi<\theta<2\pi$, respectively. Because $\cos\theta$ is an odd function with respect to the middle points of these two intervals, it follows that the contribution of such oscillators to the order parameter is zero,

$$\int_0^{2\pi} d\theta \frac{C(\omega)\cos\theta}{|\omega - \gamma\sin\theta|} = 0. \tag{11}$$

The distribution of locked oscillators is described by the delta function $\rho(\omega,\theta)=\delta(\theta-\arcsin(\omega/\gamma))$, and the order parameter is determined by

$$r = \int\int d\omega\, d\theta \cos\theta g(\omega)\delta(\theta - \arcsin(\omega/\gamma)) = \int_{-\gamma}^{\gamma} d\omega\, g(\omega)\sqrt{1 - \left(\frac{\omega}{\gamma}\right)^2}. \tag{12}$$

The contribution of all oscillators with some frequency $\omega>\gamma$ to the phase balance is

$$\int_0^{2\pi} d\theta \frac{C(\omega)\sin\theta}{|\omega - \gamma\sin\theta|} = \frac{1}{\gamma}(\omega - \sqrt{\omega^2 - \gamma^2}). \tag{13}$$

Similarly, the contribution of the oscillators with frequency $\omega<-\gamma$ is $(\omega+\sqrt{\omega^2-\gamma^2})/\gamma$. The total contribution of all drifting oscillators (those with $|\omega|>\gamma$) to the phase balance is

$$I_1 = \int_{-\infty}^{-\gamma} d\omega\, g(\omega)\frac{1}{\gamma}(\omega + \sqrt{\omega^2-\gamma^2}) + \int_{\gamma}^{\infty} d\omega\, g(\omega)\frac{1}{\gamma}(\omega - \sqrt{\omega^2-\gamma^2}). \quad (14)$$

Expressions (11) and (13) can be found also in [12].

The influence of the locked oscillators to the phase balance is

$$I_2 = \int\int d\omega\, d\theta\, \sin\theta g(\omega)\delta(\theta - \arcsin(\omega/\gamma)) = \int_{-\gamma}^{\gamma} d\omega\, g(\omega)\frac{\omega}{\gamma}. \quad (15)$$

Combining the integrals in Eqs. (14) and (15) the phase balance equation reads

$$0 = \int_{-\infty}^{\infty} d\omega\, g(\omega)\omega + \int_{-\infty}^{-\gamma} d\omega\, g(\omega)\sqrt{\omega^2-\gamma^2} - \int_{\gamma}^{\infty} d\omega\, g(\omega)\sqrt{\omega^2-\gamma^2}, \quad (16)$$

where the multiplier $1/\gamma$ has been canceled out. The first integral is

$$\int_{-\infty}^{\infty} d\omega\, g(\omega)\omega = \bar{\omega} + \omega_0, \quad (17)$$

and the phase balance equation is reduced to

$$0 = \bar{\omega} + \omega_0 + \int_{-\infty}^{-\gamma} d\omega\, g(\omega)\sqrt{\omega^2-\gamma^2} - \int_{\gamma}^{\infty} d\omega\, g(\omega)\sqrt{\omega^2-\gamma^2}. \quad (18)$$

This equation together with Eq. (12) represents a closed system for determination of the dependence on the coupling strength K of the order parameter r and the shift of the central frequency ω_0 around which the coherent oscillators become organized.

In studies of phase transitions the first thing to be found is the critical point, which in the Kuramoto model is determined by the critical coupling K_c. For the second-order phase transitions it can be calculated from Eq. (12) in the limit $\gamma = Kr \to 0$. Then it is expressed through the critical synchronizing frequency $K_c = 2/[\pi g(0)] = 2/[\pi\hat{g}(-\omega_c)]$. To obtain ω_c, the phase balance Eq. (18) should be taken in the limit $\gamma \to 0$, which means the square root can be expressed with its power-series expansion. Keeping the dominant terms in the approximate equation obtained from the expansion, the equation for the critical synchronization frequency ω_c reads

$$0 = \int_0^{\infty} d\omega \frac{g(\omega) - g(-\omega)}{\omega} = \int_0^{\infty} d\omega \frac{\hat{g}(\omega-\omega_c) - \hat{g}(-\omega-\omega_c)}{\omega}. \quad (19)$$

After the calculation of the critical synchronizing frequency ω_c for some particular distribution $\hat{g}(\omega)$, we can easily determine the critical coupling $K_c = 2/[\pi\hat{g}(-\omega_c)]$. Applying Eq. (19) one can find that ω_c is in agreement with the corresponding limiting value for the frequency obtained from the numerical solution of Eqs. (12) and (18), or formally $\omega_0 \to \omega_c$, when $K \to K_c + 0$. Since we assume that the distribution function is a normalized and well-behaved function possessing a sufficient number of derivatives, the convergence of the integral in (19) is assured at both limits.

III. EXAMPLES

As an illustration of the theory, we show in Fig. 1 solutions of the equation for the order parameter (12) and the phase balance equation (18). To continue with the analytical treatment of the model as far as possible, we have taken distribution functions for which the integrals appearing in Eqs. (12) and (18) can be expressed analytically. One such example is provided by the triangular distribution function, defined by a piecewise linear function:

$$\hat{g}_t(\omega) = \begin{cases} \hat{g}(0) - a\omega, & 0 \leq \omega \leq \hat{g}(0)/a, \\ \hat{g}(0) + b\omega, & -\hat{g}(0)/b \leq \omega < 0, \end{cases} \quad (20)$$

where a and b are positive parameters and $\hat{g}(0)$ is obtained from normalization condition $\int d\omega\, \hat{g}(\omega) = 1$. It is unimodal with maximum at $\omega = 0$. The dependence of the order parameter on the coupling strength is depicted in Fig. 1(a). There is a second-order phase transition to synchronization as is the case with symmetric distributions [3]. However, the critical value of the coupling is different from the symmetric case when $K_c = 2/[\pi\hat{g}(0)]$. It is larger and it corresponds to some value $K_c = 2/[\pi\hat{g}(-\omega_c)]$, where the critical synchronizing frequency $\omega_c = \omega_0(K_c)$ is a solution of the phase balance equation (18) in the limiting case $\gamma \to 0$ (or $r \to 0$). It means that the seed of the cluster of synchronization does not appear at the top or at the mean of the distribution of natural frequencies, which happens for the symmetric distributions, when these two points coincide, but somewhere in between. At the same time, with the change of the coupling strength, the synchronizing frequency $\omega_0(K)$ changes as well [Fig. 1(b)].

We present results for one additional distribution, a piecewise constant, which might be called "olympic." It is defined as follows:

$$\hat{g}_o(\omega) = \begin{cases} a, & -3/2 \leq \omega \leq -1/2, \\ \hat{g}(0), & -1/2 < \omega \leq 1/2, \\ b, & 1/2 < \omega \leq 3/2, \end{cases} \quad (21)$$

where again a and b are positive parameters and $\hat{g}(0)$ is constrained by normalization condition. We found a first-order phase transition for the olympic distribution, because at the critical coupling strength a macroscopic part of the oscillators from the highest plateau synchronize simultaneously. The transition to synchronization is of first order only when the synchronization seed is generated at some flat portion of the distribution, assuming that there is such a flat section. For example, for a distribution consisting of a wide nonconstant linear part and a thin adjacent constant part with height equal to the maximum of the linear segment, there is a second-order transition, because the synchronized seed appears at the nonconstant linear piece of the distribution. Flatness at the top of the distribution function appears to be a necessary,

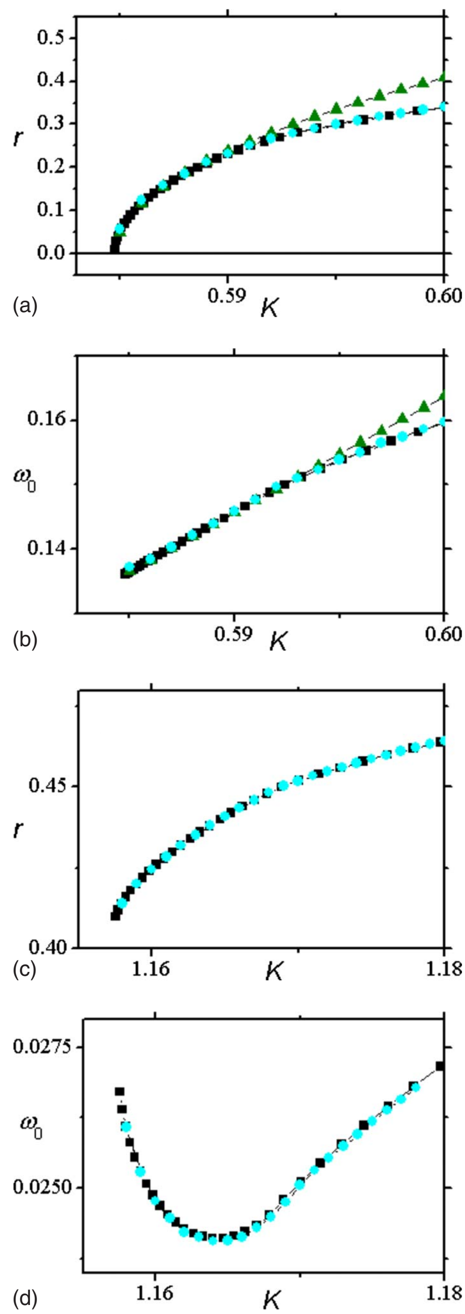

FIG. 1. (Color online) Order parameter r and frequency shift ω_0 versus coupling constant K for distributions considered in the text: (a), (b) triangular [$a=3$, $b=1$, $\hat{g}(0)=\sqrt{3/2}$], (c), (d) olympic [$a=0.25$, $b=0.2$, $\hat{g}(0)=0.55$]. The meaning of the parameters is explained in the text. Different symbols denote different results: squares, numerical solution of the system of transcendental equations (12) and (18); triangles, asymptotic relationships (29) and (30); circles, numerical solutions of the equations of motion (5).

but not sufficient condition for the existence of a first-order transition when the distribution is not symmetric.

Graphical illustrations of the numerical results for some particular choice of the parameters for the olympic distribution are provided in Fig. 1. As can be seen [Fig. 1(d)], with the increase of the coupling strength the frequency shift ω_0 can switch the direction of change. Depending on the particular case, the changes in $\omega_0(K)$, although not for the same interval of changes for the coupling parameter K, vary roughly between 10% and 20%. For larger variations of K, the variations in $\omega_0(K)$ could more than double. For infinite coupling strength the synchronizing frequency equals the mean frequency $\bar{\omega}$, as can be deduced from the phase balance equation (18) for $\gamma=Kr\to\infty$.

In addition we have analyzed the scaling laws for the order parameter and the synchronizing frequency near the critical point at $K=K_c$. For the symmetric case it is known [5] that

$$r \sim |K-K_c|^{1/m}, \tag{22}$$

where $m>0$ is a parameter describing the power-law decay of the distribution function in the vicinity of its maximum. In the case of first-order transition, it was found [9,10] that

$$|r-r_c| \sim |K-K_c|^{2/(2m+3)}, \tag{23}$$

where m is a parameter characterizing the dominant power-law term describing the tails of the symmetric distribution functions outside and in the immediate vicinity of the flat region.

For the second-order transition, we have verified from numerical evidence for the triangular distribution that the square-root law holds [4],

$$r \sim |K-K_c|^{1/2}, \tag{24}$$

with an accuracy 0.5 ± 0.01. That this is the case can be shown by expanding Eqs. (12) and (18) in the limit when $\gamma \ll 1$. First one should note that the distribution centered at the synchronizing frequency for $K=K_c$ is $g(\omega)=\hat{g}(\omega-\omega_c)$, and for $K>K_c$ it becomes $g(\omega-\delta\omega)=\hat{g}(\omega-\omega_c-\delta\omega)$ because of the drift of synchronizing frequency $\delta\omega=\omega_0-\omega_c$. Power-series expansion of the distribution for small $\delta\omega$ is

$$g(\omega-\delta\omega) \approx g(0)+g'(0)(\omega-\delta\omega)+\frac{g''(0)}{2}(\omega-\delta\omega)^2. \tag{25}$$

Then the equation for the order parameter, Eq. (12), is

$$\begin{aligned} r &= \int_{-\gamma}^{\gamma} d\omega\, g(\omega-\delta\omega)\sqrt{1-\left(\frac{\omega}{\gamma}\right)^2} \\ &\approx [g(0)-g'(0)\delta\omega]\int_{-\gamma}^{\gamma} d\omega \sqrt{1-\left(\frac{\omega}{\gamma}\right)^2} \\ &\quad + \frac{g''(0)}{2}\int_{-\gamma}^{\gamma} d\omega\, \omega^2 \sqrt{1-\left(\frac{\omega}{\gamma}\right)^2}, \end{aligned} \tag{26}$$

where only the dominant terms are kept. Near the critical point the synchronizing frequency behaves as

$$\omega_0 = \omega_c + c\gamma^2 + O(\gamma)^4, \tag{27}$$

as can be deduced from the phase balance equation (18) for $\gamma\to 0$. Then using $\delta\omega \approx cK^2r^2$ in (26) the order parameter for coupling little stronger than the critical is

$$r \approx \sqrt{\frac{16(K-K_c)}{\pi K_c^4[8c\hat{g}'(-\omega_c)-\hat{g}''(-\omega_c)]}}. \tag{28}$$

This result is also given in [4] for a more general coupling function between oscillators j and i, $\sin(\theta_j-\theta_i+\alpha)$, where α is a parameter.

For example, for the triangular distribution illustrated in Figs. 1(a) and 1(b), one finds

$$\omega_0(K) \approx \frac{\sqrt{6}}{18} + \frac{16\pi(K-K_c)}{27} \tag{29}$$

and

$$r \approx \frac{64\pi}{243}\sqrt{3\sqrt{6}\pi \ln 2}\sqrt{K-K_c}, \tag{30}$$

where $K_c=3\sqrt{6}/4\pi$. In Figs. 1(a) and 1(b), the asymptotic relations (29) and (30) for the triangular distribution are compared to the numerical solutions of Eqs. (12) and (18) and to the results found from simulations. The agreement is quite adequate. It was found numerically that the same universal square-root power law holds for all other examples we have examined [piecewise parabolic and piecewise quartic distribution functions and $g(\omega)$ consisting of two different Lorentzian distribution functions joined continuously]. The second case is interesting because for the corresponding symmetric case the exponent is $1/4$.

One may conclude that the swarm of synchronized oscillators appears by Andronov-Hopf bifurcation at the critical frequency ω_c which is located between $\bar{\omega}<0$ and the location of the maximum of the distribution function at $\omega=0$. We repeat that in the adopted notation, the synchronization occurs at the frequency $-\omega_0(K)$, but for convenience, the discussion, analytic expressions, and plots are given for $\omega_0(K)$. As the coupling constant is increased, the frequency of the synchronized oscillators generally shifts toward their mean frequency where, for unimodal distributions, there are more oscillators for recruitment into the synchronized phase. For the olympic distribution the tendency at the beginning is opposite, presumably, until all the oscillators from the most abundant central flat region are exhausted and included in the synchronized set, when their synchronized frequency starts to shift toward the mean frequency.

When the phase transition is of first order, for the critical value of the coupling K_c there is an interval of solutions for the order parameter, stretching from zero to some value r_c. It implies that the half-width of the synchronizing interval extends from zero to $\gamma_c=K_c r_c$. To different widths of the synchronizing cluster 2γ correspond different synchronizing frequencies ω_0 as required by the phase balance condition (18). From Eq. (19) one can calculate ω_c', which is related to the vanishing width of the synchronizing interval $\gamma\to 0$. The limiting value of ω_0 when the coupling decreases toward K_c is the synchronizing frequency ω_c'' when the cluster has a half-width γ_c. Thus, for the same value of the coupling K_c, there is an interval of solutions (ω_c',ω_c'') for the synchronizing frequency as is the case for the order parameter when all values in the interval $(0,r_c)$ are acceptable solutions.

We also found numerically that when the distribution is weakly asymmetric, $a\to b$ in (20), the frequency shift disappears, $\omega_c\to 0$, in accordance to the relationship $\omega_c \sim |a-b|$. This can be deduced from the equations by expanding to first order in the power of a small parameter characterizing the asymmetry of the distribution function—for example, the difference $a-b$.

As additional examples with a first-order phase transition we examined three distribution functions with flat central part and constant, linear, or quadratic tails. The characteristic exponents are close to those in the symmetric case, but the accuracy obtained is lower because it depends on the determination of the critical point (r_c,K_c) which is found only numerically. In the Appendix we show analytically that the scaling laws for the first-order transition for asymmetric functions in the neighborhood of (r_c,K_c) are actually exactly the same as for the corresponding symmetric cases and that the synchronization frequency shift is described by the same power law

$$|\omega_0(K)-\omega_0(K_c)| \sim |K-K_c|^{2/(2m+3)}. \tag{31}$$

IV. STABILITY ANALYSIS OF THE INCOHERENT PHASE

An alternative way for obtaining the critical coupling is provided by stability analysis of the Kuramoto model. A rigorous study of the linear stability of the incoherent phase for the symmetric case of the Kuramoto model was achieved by Mirollo and Strogatz [13]. Studies of the stability led to the conclusion that for couplings weaker than the critical, the incoherent phase for the original, noiseless model is neutrally stable, but when noise is added it becomes linearly stable. It was found that for symmetric distributions, the coupling at which the incoherence becomes unstable is exactly the same with the value obtained from the condition of appearance of nonzero order parameter—that is, $K_c=2/[\pi\hat{g}(0)]$. The procedure does not depend on the symmetry of distributions; it can be applied for asymmetric distribution functions as well, and we shall present its noisy version. The governing equations for the oscillator phases in the Kuramoto model with noise are

$$\dot{\hat{\theta}}_i = \hat{\omega}_i + \xi_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\hat{\theta}_j-\hat{\theta}_i), \tag{32}$$

where the noise variables $\xi_i(t)$ are characterized with zero mean and a constant D:

$$\langle \xi_i(t)\rangle = 0, \quad \langle \xi_i(t)\xi_j(t')\rangle = 2D\delta(t-t')\delta_{ij}. \tag{33}$$

The averaging is performed over different realizations of the noise. From now on in this section the carets over the phases and frequencies are dropped for convenience.

In the thermodynamic limit $N\to\infty$, the distribution $\rho(\theta,\omega,t)$ of the oscillators with frequency ω and phase θ at time t evolves according to the Fokker-Planck equation

$$\frac{\partial\rho}{\partial t} = D\frac{\partial^2\rho}{\partial\theta^2} - \frac{\partial}{\partial\theta}(\rho v), \tag{34}$$

with phase velocity

$$v(\theta,\omega,t)=\omega+Kr\sin(\psi-\theta). \tag{35}$$

The distribution of the phases at incoherence is the constant function $\rho=1/2\pi$, and to examine its linear stability one should make a small perturbation

$$\rho=\frac{1}{2\pi}+\mu(\theta,\omega)e^{\lambda t}, \tag{36}$$

where $\mu(\theta,\omega)$ is a 2π-periodic function of θ. In addition, due to the normalization condition imposed on $\rho(\theta,\omega,t)$, we have

$$\int_{-\pi}^{\pi} d\theta\,\mu(\theta,\omega)=0. \tag{37}$$

The exponent λ is generally complex and its real part α is the growth rate, determining stability, and the imaginary part β is the rotation velocity of the perturbation. After Fourier expansion of the perturbation

$$\mu(\theta,\omega)=\sum_{n=-\infty}^{\infty} b_n(\omega)e^{in\theta}, \tag{38}$$

an infinite set of equations for the Fourier coefficients follows [3]:

$$(\lambda+in\omega+n^2D)b_n=\frac{K}{2}(\delta_{n,-1}+\delta_{n,1})\int_{-\infty}^{\infty} d\omega\, b_n(\omega)\hat{g}(\omega). \tag{39}$$

One should note that the original distribution of natural frequencies $\hat{g}(\omega)$ is used, because for the incoherent phase there is no some special reference frame. In the last equation, Kronecker deltas imply that to higher-order modes correspond λ with negative real part $-n^2D$. The cases $n=\pm 1$ should be treated separately. The integral in the last equation is substituted by a constant, and the equation is solved for $b_{\pm 1}$, which are subsequently used to eliminate the introduced constant in a self-consistent manner [14]. Then the characteristic exponents λ for the first harmonics $b_{\pm 1}$ can be calculated from

$$\frac{K}{2}\int_{-\infty}^{\infty} d\nu\frac{\hat{g}(\nu)}{\lambda+D+i\nu}=1. \tag{40}$$

Expressing λ through its real and imaginary parts, $\lambda=\alpha+i\beta$, and separation of the real and imaginary parts of the integral leads to the pair of equations

$$\frac{K}{2}\int_{-\infty}^{\infty} d\nu\frac{(\alpha+D)\hat{g}(\nu)}{(\alpha+D)^2+(\beta+\nu)^2}=1,$$

$$\frac{K}{2}\int_{-\infty}^{\infty} d\nu\frac{(\beta+\nu)\hat{g}(\nu)}{(\alpha+D)^2+(\beta+\nu)^2}=0. \tag{41}$$

From the second equation one can extract the rotation velocity β at vanishing noise $D\rightarrow 0$ and at the border of stability $\alpha\rightarrow 0$. For that purpose the function appearing in the second integral can be presented by its power series

$$\frac{(\beta+\nu)\hat{g}(\nu)}{(\alpha+D)^2+(\beta+\nu)^2}\approx\frac{\hat{g}(\nu)}{\beta+\nu}\left[1-\left(\frac{\alpha+D}{\beta+\nu}\right)^2\right], \tag{42}$$

where the correction is of order $O(\alpha+D)^4$. At the limit $\alpha\rightarrow 0$, $D\rightarrow 0$, after the change of variables $\omega=\beta+\nu$, the integral for determination of the rotation velocity β is

$$\int_{-\infty}^{\infty} d\omega\frac{\hat{g}(\omega-\beta)}{\omega}=0. \tag{43}$$

Evidently, it is the same condition as the one found for determination of the frequency of the cluster of oscillators at the onset of their synchronization (19). It follows that the critical synchronizing frequency ω_c equals the perturbation rotation velocity β at the border of stability.

At the same limit $(\alpha\rightarrow 0, D\rightarrow 0)$, the integrand in the first integral in Eq. (41) behaves as a δ function,

$$\frac{(\alpha+D)\hat{g}(\nu)}{(\alpha+D)^2+(\beta+\nu)^2}\rightarrow\pi\delta(\beta+\nu)\hat{g}(\nu), \tag{44}$$

and the critical coupling is $K_c=2/[\pi\hat{g}(-\beta)]$. In this way the values of the critical frequency ω_c and the critical coupling K_c are verified with an independent procedure.

V. NUMERICAL EXPERIMENTS

In this section numerical simulations of the equations of motion (5) are used for verification of the theoretical results. The model was analytically studied for an infinite number of oscillators. In the numerical simulation we have used a population of $N=5000$ oscillators. As a numerical routine for integration of the differential equations of motion, a fourth-order Runge-Kutta method was used and the integration step was set at 0.01. The order parameter and synchronizing frequency obtained in the simulation fluctuate over time around the mean value, which settles after the transient process dies out. The values presented in Fig. 1 were calculated by time averaging over 1000 time units. The transient takes a much longer time when the system is closer to the critical point, a well-known phenomenon of critical slowing-down for phase transitions [15]. For better accuracy of the estimated order parameter a large population of units of the system is needed and this represents another restrictive condition on the numerical simulation. The synchronizing frequency is not defined below the critical point, so it was not calculated for such values of the coupling constant.

The distribution functions $\hat{g}(\omega)$ for the natural frequencies were derived from the uniform distribution $p(u)=1$, $0\leqslant u\leqslant 1$, according to the standard procedure [16]:

$$\int_0^u du\, p(u)=u=\int_{-\infty}^{\omega} d\omega'\,\hat{g}(\omega')=G(\omega). \tag{45}$$

The theory presented in this work needs an infinite number of oscillators for every frequency to exclude the influence of drifting oscillators on the order parameter. However, the simulations were performed with only one oscillator for each of the N selected frequencies from the prescribed distribution of natural frequencies $\hat{g}(\omega')$. The selection of frequencies was achieved by the above formula starting from a uniform distribution with equidistant frequencies. Nevertheless, the averages obtained for the order parameter agree very well with the corresponding theoretical quantities.

VI. CONCLUSIONS

The Kuramoto mean-field model for synchronization of oscillators has been studied for the case of an asymmetric unimodal distribution of natural frequencies which should be more common than the special cases requiring symmetry. Introducing a phase balance condition [Eq. (18)] arising from the main idea that the initial cluster of synchronized oscillators has a frequency which is different from the most probable frequency for the distribution, a stationary solution to the problem has been obtained for several examples. The equations are transcendental and had to be solved numerically.

The numerical evidence obtained from their solutions is in agreement with the analytically derived universal square-root power law for the order parameter as a function of the coupling strength near the critical point, independently of the character of the maximum in the distribution function. First-order phase transitions are found only for distributions with a flat section when the seed of the synchronizing cluster is born within the flat part. The singular behavior of the order parameter depends on the shape of the adjacent tails of the distribution function outside of the flat region creating the initial synchronized set. Similarly to previous results [10], the first-order transition lacks metastability and hysteresis. Again we find agreement between the numerically estimated scaling laws near the critical coupling and those predicted analytically for first-order transitions. The results were further corroborated by numerical solutions of the equations of motion for an ensemble of oscillators and also from the study of the stability of the incoherent phase. The stability analysis has shown that the imaginary part of the exponential growth factor at the onset of instability of the incoherent state is identical to the synchronization frequency at the critical point.

The dependence of the synchronizing frequency on the coupling strength may have some significance because it provides a way for the system to adapt to the changes in the environment if we assume that such changes are represented by an effective coupling constant. Further exploration could follow the recently suggested model [17] in which the effective interaction parameter is taken to be dependent on the order parameter r as Kr^z, where z is a parameter. The shift of the frequency of the coherent phase should be easily measurable in an experiment, and it will serve as an indication of the asymmetry of the distribution of natural frequencies.

APPENDIX: DERIVATION OF THE CHARACTERISTIC EXPONENTS FOR THE FIRST-ORDER PHASE TRANSITIONS

The self-consistency equation (12) for the order parameter r, expressed through the phase θ, is

$$r = Kr\int_{-\pi/2}^{\pi/2} d\theta \cos^2 \theta g(Kr \sin \theta), \tag{A1}$$

where the integration is made only over the interval of stable solutions of the phases $|\theta| < \pi/2$. The first-order phase transition is due to the existence of a dominant flat top of the distribution. When the synchronized cluster comprises only oscillators from the flat section, the equation for r is independent of the width of the cluster because the distribution is constant in that part,

$$r = Krg_0\int_{-\pi/2}^{\pi/2} d\theta \cos^2 \theta, \tag{A2}$$

where $g_0 = g(0)$, and the equation is consistent only for the critical value of the coupling $K_c = 2/(\pi g_0)$. The order parameter r varies according to the width of the cluster [see Eq. (9)],

$$r = \gamma/K_c, \tag{A3}$$

and its critical value r_c is attained when one of the ends of the synchronized interval reaches one of the borders of the plateau, in our case the left one, because the mean of the distribution $\hat{g}(\omega)$ is negative. Then the half-width of the synchronized cluster is $\gamma_c = K_c r_c$.

When the coupling is little larger than critical, other oscillators join the cluster. From the right side of the cluster they are from the plateau, and from the left side they belong to the part of the distribution that differs from the constant g_0. Also, there is a shift of the locking frequency of the cluster, $-\omega_0$, in order to satisfy the phase balance equation (18). Numerical analysis shows that for distribution functions with a flat top and negative mean, $-\omega_0$ drifts toward positive frequencies and becomes $-(\omega_0 - \delta\omega)$. However, the direction of the drift is irrelevant for the conclusions. In the reference frame of the new synchronizing frequency, the left border of the plateau attains a value $-\omega_b = -(\gamma_c + \delta\omega)$.

For couplings a little stronger than critical, near the left boundary of the plateau the distribution has a general form

$$g(\omega) = g_0 - C|\omega + \omega_b|^m H(-\omega - \omega_b), \tag{A4}$$

where C is a positive constant, $m > 0$ is a parameter, and $H(\omega)$ is the unit-step Heaviside function. Introducing the distribution (A4) in Eq. (A1), the order parameter is given by

$$r = Krg_0\int_{-\pi/2}^{\pi/2} d\theta \cos^2 \theta - CKr\int_{-\pi/2}^{\theta_0} d\theta \cos^2 \theta (Kr)^m|\sin \theta + \sin \theta_0|^m, \tag{A5}$$

where the phase θ_0 corresponds to the left border of the plateau $-\omega_b$ in accordance with Eq. (9). The phase θ_0 deviates from $-\pi/2$ as $\delta\theta = \theta_0 + \pi/2$. With suitable changes of variables and taking dominant terms in power series of the trigonometric functions, it can be shown that the second integral in Eq. (A5) scales as $\delta\theta^{2m+3}$ [10]. Taking $K = K_c + \delta K$ one obtains

$$1 = (K_c + \delta K)\frac{\pi g_0}{2} - A\,\delta\theta^{2m+3}, \tag{A6}$$

where A is a constant. Using the critical value of the coupling, $K_c=2/(\pi g_0)$, the following scaling relationship is obtained:

$$\delta K \sim \delta\theta^{2m+3}. \tag{A7}$$

The phase θ_0 corresponding to the left border of the plateau $-\omega_b$ is defined with [see Eq. (9)]

$$Kr \sin \theta_0 = -(\gamma_c + \delta\omega). \tag{A8}$$

Expanding near the critical point,

$$\delta K = K - K_c, \quad \delta r = r - r_c, \quad \delta\theta = \theta_0 + \pi/2, \tag{A9}$$

one can easily get another equation relating the variations δr, δK, and $\delta\omega$,

$$0 = K_c \delta r + r_c \delta K + \frac{K_c r_c}{2}\delta\theta^2 - \delta\omega = K_c \delta r + B\,\delta K^{2/(2m+3)} - \delta\omega, \tag{A10}$$

where only the dominant term with δK dependence is kept and B is a constant.

Now we will analyze the behavior of the frequency shift $\delta\omega$ near the critical point. At the critical point the phase balance equation (18) is

$$0 = \bar{\omega} + \omega_c + \int_{-\infty}^{-\gamma_c} d\omega\, g_c(\omega)\sqrt{\omega^2 - \gamma_c^2} - \int_{\gamma_c}^{\infty} d\omega\, g_c(\omega)\sqrt{\omega^2 - \gamma_c^2}, \tag{A11}$$

where we have used the distribution g_c centered at the critical synchronizing frequency ω_c. Then for coupling stronger than the critical, the phase balance equation expressed through g_c is

$$0 = \bar{\omega} + \omega_c - \delta\omega + \int_{-\infty}^{-\gamma_c-\delta\gamma} d\omega\, g_c(\omega + \delta\omega)\sqrt{\omega^2 - (\gamma_c + \delta\gamma)^2} - \int_{\gamma_c+\delta\gamma}^{\infty} d\omega\, g_c(\omega + \delta\omega)\sqrt{\omega^2 - (\gamma_c + \delta\gamma)^2}. \tag{A12}$$

Let us denote by D^- (D^+) the differences between the first (second) integrals in Eqs. (A11) and (A12). The first difference is

$$D^- = \int_{-\infty}^{-\gamma_c-\delta\gamma} d\omega\, g_c(\omega + \delta\omega)\sqrt{\omega^2 - (\gamma_c + \delta\gamma)^2} - \int_{-\infty}^{-\gamma_c} d\omega\, g_c(\omega)\sqrt{\omega^2 - \gamma_c^2}. \tag{A13}$$

Omitting the higher-order terms of $\delta\omega$ and $\delta\gamma$, it is reduced to

$$D^- = -\int_{-\gamma_c-\delta\gamma}^{-\gamma_c} d\omega\, g_c(\omega)\sqrt{\omega^2 - \gamma_c^2} + \delta\omega\int_{-\infty}^{-\gamma_c} d\omega \frac{\partial g_c}{\partial\omega}\sqrt{\omega^2 - \gamma_c^2} - \delta\gamma\int_{-\infty}^{-\gamma_c} d\omega \frac{g_c(\omega)\gamma_c}{\sqrt{\omega^2 - \gamma_c^2}}. \tag{A14}$$

The third integral in D^- is a constant, and integration by parts shows that the same is true for the second integral. The first integral is of the order $\delta\gamma^{3/2}$. Therefore,

$$D^- = C\,\delta\gamma + D\,\delta\omega, \tag{A15}$$

where C and D are constants. In the same manner it can be shown that $D^+=E\,\delta\gamma+F\,\delta\omega$ with E and F constants. Subtraction of Eqs. (A11) and (A12) gives

$$0 = D^+ + D^- - \delta\omega, \tag{A16}$$

from where it is clear that

$$\delta\omega \sim \delta\gamma. \tag{A17}$$

Since $\gamma=Kr$, near the critical point $\delta\gamma=K_c\delta r+r_c\delta K$ and

$$\delta\omega \sim K_c\delta r + r_c\delta K. \tag{A18}$$

Finally using Eq. (A18) in Eq. (A10) and after neglecting terms proportional to δK in comparison to $\delta K^{2/(2m+3)}$,

$$\delta r \sim \delta K^{2/(2m+3)}. \tag{A19}$$

The value of the characteristic exponent $2/(2m+3)$ is the same as for the symmetric case [10]. In addition, from the last two equations it follows that

$$\delta\omega \sim \delta K^{2/(2m+3)}. \tag{A20}$$

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